

Equivariant algorithms for constraint satisfaction problems over coset templates[☆]

Sławomir Lasota

University of Warsaw

Abstract

We investigate the Constraint Satisfaction Problem (CSP) over templates with a group structure, and algorithms solving CSP that are *equivariant*, i.e. invariant under a natural group action induced by a template. Our main result is a method of proving the implication: if CSP over a coset template T is solvable by a local equivariant algorithm then T is 2-Helly (or equivalently, has a majority polymorphism). Therefore bounded width, and definability in fixed-point logics, coincide with 2-Helly. Even if these facts may be derived from already known results, our new proof method has two advantages. First, the proof is short, self-contained, and completely avoids referring to the omitting-types theorems. Second, it brings to light some new connections between CSP theory and descriptive complexity theory, via a construction generalizing CFI graphs.

Keywords:

1. Introduction

Many natural computational problems may be seen as instantiations of a generic framework called *constraint satisfaction problems* (CSP). In a nutshell, a CSP is parametrized by a *template*, a finite relational structure T ; the CSP over T asks if a given relational structure I over the same vocabulary as T admits a homomorphism to T (called a solution of I). For every template T , the CSP over T (denoted $\text{CSP}(T)$) is always in NP; a famous conjecture due to Feder and Vardi [14] says that for every template T , the $\text{CSP}(T)$ is either solvable in P, or NP-complete.

We concentrate on *coset templates* where, roughly speaking, both the carrier set and the relations have a group structure. The coset templates are cores and admit a Malcev polymorphism, and are thus in P [13, 7]. A coset template T naturally induces a group action on (partial) solutions. If, roughly speaking, the induced group action can be extended to the state space of an algorithm solving $\text{CSP}(T)$, and the algorithm execution is invariant under the group action, we call the algorithm *equivariant*. We investigate equivariant algorithms which are *local*, i.e. update only a bounded amount of data in every single step of execution.

A widely studied family of local equivariant algorithms is the *local consistency algorithms* that compute families of partial solutions of bounded size conforming to a local consistency condition. Templates T whose $\text{CSP}(T)$ is solvable by a local consistency algorithm are said to have *bounded width*. Another source of examples of local equivariant algorithms are logics (via their decision procedures); relevant logics for us will be fix-point extensions of first order logic, like LFP or IFP or IFP+C (IFP with counting quantifiers) [12]. We say that $\text{CSP}(T)$ is de-

finable in a logic if some formula of the logic defines the set of all solvable instances of $\text{CSP}(T)$.

Our technical contribution is the proof of the following implication: if $\text{CSP}(T)$, for a coset template T , is solvable by a local equivariant algorithm then T is 2-Helly. In consequence, all local equivariant algorithms that can capture 2-Helly templates are equally expressive. The 2-Helly property says that for every partial solution h of an instance I , if h does not extend to a solution of I then the restriction of h to some two elements of its domain does not either. This is a robust property of templates with many equivalent characterizations (e.g. strict width 2, or existence of a majority polymorphism) [14]. As a corollary we obtain equivalence of the following conditions for coset templates: (i) 2-Helly; (ii) bounded width; and (iii) definability in fix-point extensions of first-order logic. The corollary is not a new result; equivalence of the first two conditions may be inferred e.g. from Lemma 9 in [11] (even for all core templates with a Malcev polymorphism), while equivalence of the last two ones follows from [1] together with the results of [3] (cf. also [4]). All these results build on Tame Congruence Theory [15], and their proofs are a detour through the deep omitting-type theorems, cf. [16]. Contrarily to this, our proof has an advantage of being short, elementary, and self-contained, thus offering a direct insight into the problem.

Finally, our proof brings to light interesting connections between the CSP theory and the descriptive complexity theory: the crucial step of the proof is essentially based on a construction similar to *CFI graphs*, the intricate construction of Cai, Fürer and Immerman [8]. CFI graphs have been designed to separate properties of relational structures decidable in polynomial time from IFP+C. A similar construction has been used later in [6] to show lack of determination of Turing machines in sets with atoms [5]. The crucial step of our proof is actually a significant generalization of the construction of [6].

[☆]Supported by the NCN grant 2012/07/B/ST6/01497.

For completeness we mention a recent paper of Barto [2] which announces the collapse of bounded width hierarchy for *all* templates: bounded width implies width (2, 3), which is however weaker than 2-Helly in general.

2. Preliminaries

2.1. Constraint satisfaction problems

A *template* T is a finite relational structure, i.e. consists of a finite carrier set T (denoted by the same symbol as a template) and a finite family of relations in T . Each relation $R \subseteq T^n$ is of a specified arity, $\text{arity}(R) = n$. Let T be fixed henceforth.

An instance I over a template T consists of a finite set I of elements, and a finite set of *constraints*. A constraint, written $R(a_1, \dots, a_n)$, is specified by a template relation R and an n -tuple of elements of I , where $\text{arity}(R) = n$.

A partial function h from I to T , with $\{a_1, \dots, a_n\} \subseteq \text{dom}(h)$, *satisfies* a constraint $R(a_1, \dots, a_n)$ in I when $R(h(a_1), \dots, h(a_n))$ holds in T . If h satisfies all constraints in its domain, h is a *partial solution* of I , and h is a *solution* when it is total. By the size of a partial solution h we mean the size of $\text{dom}(h)$. The constraint satisfaction problem over T , denoted $\text{CSP}(T)$, is a decision problem that asks if a given instance over T has a solution.

There are many equivalent formulations of the problem. For instance, one can see I and T as relational structures over the same vocabulary, and then $\text{CSP}(T)$ asks if there is a homomorphism from I to T .

2.2. 2-Helly templates

For an instance I over some template, and $k < j$, a (k, j) -*anomaly* is a partial solution h of I of size j that does not extend to a solution, such that restriction of h to every k -element subset of $\text{dom}(h)$ does extend to a solution. Clearly a (k, j) -anomaly is also (k', j) -anomaly, for $k' < k$.

Definition 2.1. A template T is *2-Helly* if no instance of T admits a $(2, j)$ -anomaly, for $j > 2$.

In other words: for every partial solution h of size $j > 2$, if the restriction of h to every 2-element subset of its domain extends to a solution then h does extend to a solution too. Analogously one may define k -Helly for arbitrary k , which however will not be needed here.

We conveniently characterize 2-Helly templates as follows.

Lemma 2.2. A template T is 2-Helly iff no instance of T admits a $(k, k+1)$ -anomaly, for $k \geq 2$.

Proof. For one direction, we observe that a $(k, k+1)$ -anomaly is also a $(2, k+1)$ -anomaly.

For the other direction, consider an instance with some fixed $(2, j)$ -anomaly h , for $j > 2$. For every subset $X \subseteq \text{dom}(h)$, the restriction $h|_X$ either extends to a solution of I , or not. Consider the minimal subset X wrt. inclusion such that $h|_X$ does not extend to a solution of I . For all strict subsets $X' \subseteq X$, $f|_{X'}$ extends to a solution, hence $f|_X$ is a $(k-1, k)$ -anomaly, where k is the size of X . Note that $k > 2$. \square

2.3. The pp-definable relations

We adopt the convention to mention explicitly the free variables of a formula $\phi(x_1, \dots, x_n)$. In the specific instances I of $\text{CSP}(T)$ used in our proof it will be convenient to use *pp-definable* relations, i.e. relations definable by an existential first-order formula of the form:

$$\phi(x_1, \dots, x_n) \equiv \exists x_{n+1}, \dots, x_{n+m}. \psi_1 \wedge \dots \wedge \psi_l, \quad (1)$$

where every subformula ψ_i is an atomic proposition $R(x_{i_1}, \dots, x_{i_j})$, for some template relation R . The formula ϕ defines the n -ary relation in T containing the tuples

$$(t_1, \dots, t_n) \in T^n$$

such that the valuation $x_1 \mapsto t_1, \dots, x_n \mapsto t_n$ satisfies ϕ . The pp-definable relations are closed under projection and intersection.

In the sequel we feel free to implicitly assume that elements of an instance are totally ordered. The implicit order allows us to treat (partial) solutions as tuples, and allows to state the following useful fact:

Fact 2.3. Let $X \subseteq I$ be a subset of an instance. The set of partial solutions with domain X that extend to a solution of I , if nonempty, is pp-definable.

2.4. Almost-direct product of groups

Overloading the notation, we write 1 for the identity element in any group. We use the diagrammatic order for writing the group operation $\tau\pi$ on elements τ, π of a group.

In the proof we will need the following elementary notion from group theory.¹

Definition 2.4. Let G_1, G_2 and G_3 be arbitrary finite groups and let $H \leq G_1 \times G_2 \times G_3$ be a subgroup of the direct product. We call H an *almost-direct product* of G_1, G_2, G_3 if H verifies the following conditions:

$$H \neq G_1 \times G_2 \times G_3 \quad (2)$$

$$\forall \pi_2 \in G_2, \pi_3 \in G_3, \exists \pi_1 \in G_1. (\pi_1, \pi_2, \pi_3) \in H \quad (3)$$

$$\forall \pi_1 \in G_1, \pi_3 \in G_3, \exists \pi_2 \in G_2. (\pi_1, \pi_2, \pi_3) \in H \quad (4)$$

$$\forall \pi_1 \in G_1, \pi_2 \in G_2, \exists \pi_3 \in G_3. (\pi_1, \pi_2, \pi_3) \in H \quad (5)$$

Furthermore, an almost-direct product H is strict if π_1 (resp. π_2, π_3) in condition (3) (resp. (4), (5)) is uniquely determined.

Let $H \leq G_1 \times G_2 \times G_3$ be an almost-direct product. Consider the following normal subgroup N_1 of G_1 :

$$N_1 = \{\pi_1 \in G_1 : (\pi_1, 1, 1) \in H\}.$$

Likewise define the normal subgroups N_2 and N_3 of G_2 and G_3 , respectively. In consequence, the product $N = N_1 \times N_2 \times N_3$ is a normal subgroup of H . Define the groups $[G_1], [G_2], [G_3]$ and $[H]$ as the quotients by N_1, N_2, N_3 and N , respectively.

¹ The notion seems to be of independent interest; it is related to the *arity* of a permutation group, as investigated for instance by Cherlin et al. in [9].

By the definition of N_1 , the quotient group $[G_1]$ is obtained by identifying its elements π_1, π'_1 that are equivalent:

$$\pi_1 \equiv \pi'_1 \Leftrightarrow (\forall \pi_2, \pi_3, (\pi_1, \pi_2, \pi_3) \in H \Leftrightarrow (\pi'_1, \pi_2, \pi_3) \in H).$$

Similarly one defines the equivalences \equiv_2 and \equiv_3 . Note that H is closed under the three equivalences; for instance,

$$(\pi_1, \pi_2, \pi_3) \in H \text{ and } \pi_1 \equiv \pi'_1 \implies (\pi'_1, \pi_2, \pi_3) \in H. \quad (6)$$

Lemma 2.5. *The quotient group $[H]$ is a strict almost-direct product of $[G_1], [G_2], [G_3]$.*

Proof. $[H]$, being the quotient of H , is an almost-direct product of $[G_1], [G_2], [G_3]$. We claim that $[H]$ is strict. Concentrating on point (3) in Definition 2.4 (the remaining two conditions are treated similarly), we need to prove uniqueness of π_1 . Suppose

$$(\pi_1, \pi_2, \pi_3) \in [H] \quad \text{and} \quad (\pi'_1, \pi_2, \pi_3) \in [H];$$

we need to derive $\pi_1 = \pi'_1$. As H is closed under the three equivalences, there are some $\rho_1, \rho'_1, \rho_2, \rho_3$ such that

$$\tau = (\rho_1, \rho_2, \rho_3) \in H, \quad \tau' = (\rho'_1, \rho_2, \rho_3) \in H,$$

and (writing $[\rho]$ for the equivalence class containing ρ)

$$[\rho_1] = \pi_1, \quad [\rho'_1] = \pi'_1, \quad [\rho_2] = \pi_2, \quad [\rho_3] = \pi_3;$$

and we need to derive $\rho_1 \equiv \rho'_1$. The equivalence follows easily: whenever $\sigma = (\pi_1, \tau_2, \tau_3) \in H$, we have

$$(\pi'_1, \tau_2, \tau_3) = \sigma \tau^{-1} \tau' \in H.$$

□

Lemma 2.6. *A strict almost-direct product is commutative.*

Proof. Let $H \leq G_1 \times G_2 \times G_3$ be a strict almost-direct product and let $\pi, \tau \in G_1$. We know that there exist $\rho_2 \in G_2$ and $\rho_3 \in G_3$ so that (we do not use the uniqueness of ρ_2 and ρ_3 here):

$$(\pi, 1, \rho_3) \in H \quad \text{and} \quad (\tau, \rho_2, 1) \in H.$$

Applying the group operation to these two elements in two different orders we get:

$$(\pi\tau, \rho_2, \rho_3) \in H \quad \text{and} \quad (\tau\pi, \rho_2, \rho_3) \in H.$$

Now using the uniqueness of $\pi\tau$ (and $\tau\pi$), we deduce that $\pi\tau = \tau\pi$. As π and τ have been chosen arbitrarily, the group G_1 is commutative. Likewise for G_2 and G_3 , and in consequence also for the subgroup $H \leq G_1 \times G_2 \times G_3$. □

3. Coset templates

Below by a coset we always mean a right coset. (This choice is however arbitrary and we could consider left cosets instead.)

Definition 3.1. *Coset templates are particular templates T that satisfy the following conditions:*

- *the carrier set of T is a disjoint union of groups, call these groups carrier groups;*
- *every n -ary relation R in T is a coset in the direct product $G_1 \times \dots \times G_n$ of some carrier groups G_1, \dots, G_n ;*
- *for a relation $R \subseteq G_1 \times \dots \times G_n$ in T and $\pi \in G_1 \times \dots \times G_n$, the coset $R\pi$ is also a relation in T ;*
- *for every carrier group G , T has a unary relation $\{1\}$ containing exactly one element, the identity of G .*

Note that the last two conditions imply that a coset template contains every singleton as a unary relation, and thus is a rigid core, i.e. admits no nontrivial endomorphisms.

Example 3.1. Here is a family of coset templates T_n , for $n \geq 2$. The carrier set of T_n is $\{1, \pi\}$, the cyclic group of order 2. Relations of T_n are, except the two singleton unary relations $1(_)$ and $\pi(_)$, two n -ary relations

$$R_{\text{even}}, R_{\text{odd}} \subseteq T^n$$

containing n -tuples where π appears an even (resp. odd) number of times. Template T_2 is 2-Helly, while for $n > 2$, template T_n is not. Indeed, a (2,3)-anomaly is admitted by an instance over T_3 , consisting of three elements a_1, a_2, a_3 and four constraints:

$$\pi(a_1) \quad \pi(a_2) \quad \pi(a_3) \quad R_{\text{even}}(a_1, a_2, a_3).$$

Consider a relation $R \subseteq G_1 \times \dots \times G_n$ in a coset template, and an instance I . For a constraint $R(a_1, \dots, a_n)$ in I and $i \in \{1 \dots n\}$, we call G_i a *constraining group* of a_i . In order to have a solution, an instance I has to be non-contradictory, in the sense that every element must have exactly one constraining group (elements with no constraining group may be safely removed from I). We only consider non-contradictory instances from now on.

Consider a fixed coset template T and an instance I over T . By a *pre-solution* of I we mean any function $s : I \rightarrow T$ that maps every element $i \in I$ to an element of the constraining group of i . Pre-solutions of an instance I form a group, with group operation defined point-wise. One can also speak of partial pre-solutions, whose domain is a subset of I . Using an implicit order of elements of an instance, (partial) pre-solutions of I are elements of the direct product of constraining groups of (some) elements of I .

We distinguish *subgroup instances*, where all relations R appearing in the constraints $R(a_1, \dots, a_n)$ are subgroups, instead of arbitrary cosets.

Fact 3.2. (1) *The set of all solutions \mathcal{H} of an instance I , if nonempty, is a coset in the group of pre-solutions.* (2) *In consequence, if I is a subgroup instance then \mathcal{H} is a subgroup of the group of pre-solutions.*

Proof. To show (1) observe that for every constraint $c = R(a_1, \dots, a_n)$ in I , the set of all pre-solutions \mathcal{H}_c satisfying that particular constraint is a coset in the group of pre-solutions. As solutions \mathcal{H} are exactly the intersection,

$$\mathcal{H} = \bigcap_c \mathcal{H}_c,$$

for c ranging over all constraints in I , by closure of cosets under nonempty intersection we derive that \mathcal{H} is a coset.

(2) follows by an observation that the tuple $(1, \dots, 1)$ of identities is always a solution, in case of a subgroup instance. \square

Every pp-definable relation is essentially a projection of the set of solutions of some instance (variables are element of the instance, and atomic propositions are its constraints), and by Fact 3.2 we derive the following corollary:

Fact 3.3. (1) Every pp-definable relation $R \subseteq G_1 \times \dots \times G_n$ in T is a coset in $G_1 \times \dots \times G_n$. (2) If R is pp-definable and $\pi \in G_1 \times \dots \times G_n$ then $R\pi$ is pp-definable as well.

We will later exploit the following property of coset templates:

Lemma 3.4. If some subgroup instance admits a $(k, k+1)$ -anomaly, for $k \geq 2$, then some subgroup instance admits a $(k-1, k)$ -anomaly.

Proof. Fix a $(k, k+1)$ -anomaly h in a subgroup instance I , for some $k \geq 2$, and choose an arbitrary element $a \in \text{dom}(h)$. Let $X = \text{dom}(h) \setminus \{a\}$. Define the new instance I' , with the same domain as I , whose constraints are all constraints of I plus one additional unary constraint $1(a)$ requiring that a should be mapped to the identity in its constraining group.

As h is an anomaly, the restriction $h|_{\{a\}}$ extends to a solution of I , i.e. I has a solution \bar{h} satisfying $\bar{h}(a) = h(a)$. Using an arbitrary such solution we define another partial solution h' of I with $\text{dom}(h') = \text{dom}(h) = X \cup \{a\}$:

$$h'(x) = h(x) \cdot \bar{h}^{-1}(x), \quad \text{for } x \in X \cup \{a\}.$$

Consider the restriction $h'' = h'|_X$. We claim that h'' is a $(k-1, k)$ -anomaly in I' . Indeed, for every subset $X' \subseteq X$ of size $k-1$, $h|_{X' \cup \{a\}}$ extends to a solution of I , hence $h'|_{X' \cup \{a\}}$ extends to a solution of I' , and hence $h''|_{X'}$ also extends to a solution of I' . Moreover, h does not extend to a solution of I , hence h' does not extend to a solution of I' , and thus h'' also does not extend to a solution of I' , as every solution of I' is forced to map a to 1. \square

3.1. Action of pre-solutions

For a fixed instance I , define the action of pre-solutions on (partial) pre-solutions (thus in particular on (partial) solutions). For a (partial) pre-solution $h : I \rightarrow T$ and a pre-solution s , let $h \cdot s$ be defined by the point-wise group operation:

$$(h \cdot s)(a) = h(a) s(a), \quad \text{for } a \in \text{dom}(h).$$

We will apply the action to the instance I itself: let $I \cdot s$ be an instance with the same carrier set as I , whose constraints are obtained from the constraints of I as follows: for every constraint $R(a_1, \dots, a_n)$ of I , the instance $I \cdot s$ contains a constraint

$$(R\pi)(a_1, \dots, a_n), \quad \text{where } \pi = (s(a_1), \dots, s(a_n)).$$

Note that the action preserves constraining groups, and hence pre-solutions, of an instance.

It is important to notice that solvability is invariant under the action of pre-solutions:

Fact 3.5. If h is a solution of I then $h \cdot s$ is a solution of $I \cdot s$.

4. Local equivariant algorithms

In the following we consider deterministic algorithms which run in *stages*, and in every i th stage a new object $\mathcal{D}_i(I)$ is computed as a function of the instance I and previously computed objects $\mathcal{D}_1(I), \dots, \mathcal{D}_{i-1}(I)$. Thus an execution of an algorithm can be described as a sequence of $n(I)$ objects

$$\mathcal{D}_1(I), \mathcal{D}_2(I), \dots, \mathcal{D}_{n(I)}(I).$$

The outcome of an algorithm is determined by the final object $\mathcal{D}_{n(I)}(I)$.

We assume some action of pre-solutions s of I on the objects $\mathcal{D}_i(I)$, written $\mathcal{D}_i(I) \cdot s$. An algorithm is called *equivariant* when it commutes with the action: for every pre-solution s ,

$$n(I \cdot s) = n(I)$$

$$\mathcal{D}_i(I \cdot s) = \mathcal{D}_i(I) \cdot s, \quad \text{for every } i \leq n(I).$$

We say that s *fixes* $\mathcal{S} \subseteq I$ when $s(x) = 1$ for all $x \in \mathcal{S}$. An equivariant algorithm is *local* if there is a locality bound $d \in \mathbb{N}$ independent from an instance, such that for every instance I and $i \leq n(I)$, either (L0) holds, or both (L1) and (L2) hold:

(L0) There is a subset $\mathcal{S} \subseteq I$ of size at most d such that $\mathcal{D}_i(I)$ depends only on the restriction of I to \mathcal{S} .

(L1) $\mathcal{D}_i(I)$ depends only on at most d objects among $\mathcal{D}_1(I), \dots, \mathcal{D}_{i-1}(I)$.

(L2) There is a subset $\mathcal{S} \subseteq I$ of size at most d such that $\mathcal{D}_i(I) \cdot s = \mathcal{D}_i(I)$ for every pre-solution s fixing \mathcal{S} .

The last condition (L2) is motivated by sets with atoms [5, 6] – it corresponds to bounded support. Roughly speaking, (L2) says that $\mathcal{D}_i(I)$ is only related to a bounded number of elements of I . For illustration, we demonstrate that local consistency algorithms, as well as the decision procedures of fix-point extensions of first-order logic, are equivariant and local.

4.1. Local consistency algorithms

Consider an instance I and a family \mathcal{H} of its partial solutions of size at most k , for some $k > 0$. It will be convenient to split \mathcal{H} into the subfamilies \mathcal{H}_X , where $\mathcal{H}_X = \{h \in \mathcal{H} : \text{dom}(h) = X\}$. Fix k and $l \geq k$, and consider two subsets $X \subseteq Y$ of I of size k and l , respectively. A partial solution $h \in \mathcal{H}_X$ is *consistent* wrt. \mathcal{H} and (X, Y) if

h extends to a partial solution h' with $\text{dom}(h') = Y$, whose restriction $h'|_{X'}$ to every subset $X' \subseteq Y$ of size at most k belongs to $\mathcal{H}_{X'}$.

The (k, l) -consistency algorithm takes as input an instance I over T , and computes the greatest family \mathcal{H} of partial solutions of size at most k , such that every $h \in \mathcal{H}_X$ is consistent wrt. \mathcal{H} and (X, Y) , for every X and Y as above. The algorithm starts with \mathcal{H} containing all partial solutions of size k , and proceeds by iteratively removing from \mathcal{H} all partial solutions h that falsify the consistency condition. The order of removing

is irrelevant, but in order to guarantee equivariance we assume some fixed enumeration $(X_1, Y_1), \dots, (X_n, Y_n)$ of all pairs (X, Y) of subsets of I as above, that only depends on the size of the input I , but not on the constraints in I , and that the algorithm proceeds by iteratively executing the following subroutine until stabilization:

for $i = 1, 2, \dots, n$,

$$\mathcal{H}_{X_i} := \{h \in \mathcal{H}_{X_i} : h \text{ is consistent wrt. } H \text{ and } (X_i, Y_i)\} \quad (*)$$

Every single update $(*)$ of \mathcal{H}_{X_i} , for some pair (X_i, Y_i) , constitutes a stage of the algorithm.

When the stabilization is reached and \mathcal{H} is nonempty then all the subfamilies \mathcal{H}_X are also nonempty. The (k, l) -consistency algorithm accepts if the family \mathcal{H} computed by the algorithm is nonempty; otherwise, the algorithm rejects.

Lift the action of pre-solutions to subfamilies \mathcal{H}_X by direct image: $\mathcal{H}_X \cdot s = \{h \cdot s : h \in \mathcal{H}_X\}$. The (k, l) -consistency algorithm is equivariant then: writing $\mathcal{H}_X(I, i)$ for the value of \mathcal{H}_X after the i th stage of execution on an instance I , we have:

$$\mathcal{H}_X(I \cdot s, i) = \mathcal{H}_X(I, i) \cdot s,$$

for every I , every its pre-solution s , every $X \subseteq I$ and every i . Furthermore, the (k, l) -consistency algorithm can be easily turned into a local one. Initially, subfamilies \mathcal{H}_X satisfy (L0) for $d = k$. Otherwise, (L1) clearly holds, as the new value of \mathcal{H}_{X_i} computed in one stage $(*)$ depends only on

$$d = \binom{l}{k} + \binom{l}{k-1} + \dots + \binom{l}{1}$$

values of $\mathcal{H}_{X'}$, for subsets $X' \subseteq Y_i$ of size at most k . (L2) holds too: $\mathcal{H}_X(I, i) \cdot s = \mathcal{H}_X(I, i)$ for all pre-solutions fixing X .

4.2. Fix-point logics

There are many fix-point extensions of first-order logic. The logic LFP offers a construct of the least fix-point of a function definable by a formula. Here is an example formula:

$$\phi(u, v) \equiv \text{LFP}_{R, x, y}[E(x, y) \vee \exists z (E(x, z) \wedge R(z, y))](u, v).$$

The formula has two free variables u, v and defines the transitive closure of a binary relation E . As a further example, the formula $\forall x, y \phi(x, y)$ defines strong connectedness.

Evaluation of a formula of the form $\text{LFP}_{R, \vec{x}}$ amounts to an iterative computation of the set of valuations of the variables \vec{x} , starting from the empty set of valuations, until stabilization. Given an arbitrary LFP formula ϕ , a set of valuations is to be computed for every subformula of ϕ . This can be turned into a stage-based local algorithm as follows. Let ϕ be a fixed LFP formula. The algorithm computes the sets \mathcal{H}_X , indexed by finite tuples $X \in I^*$ of elements of an instance I , such that for each $X = (a_1, \dots, a_n)$, the set \mathcal{H}_X contains a set of subformulas ψ of ϕ for which $\psi(a_1, \dots, a_n)$ holds. The length of tuples X is bounded by the greatest number of free variables of a subformula ϕ ; and every update of a set \mathcal{H}_X only depends on a

bounded number of other sets. Therefore, the decision procedure for ϕ is local. It is also equivariant, as renaming the relations in I into relations in $I \cdot s$ does not affect the iterative computation.

In the similar vein one argues that more expressive logics, like IFP (where the computation of fix-points is performed in the inflationary manner) or IFP+C (extension of IFP with counting), yield local and equivariant decision procedures as well.

4.3. Non-equivariant algorithms

As expected, many algorithms fail to satisfy equivariance. As a first example, consider a naive ineffective algorithm that enumerates all pre-solutions h and tests each for being a solution. Enumerating and processing pre-solutions can be performed element-wise, thus leaving a hope for locality. However, equivariance is violated. Indeed, suppose that on an instance I , the values $h(x)$ of pre-solution h for an element $x \in I$ are enumerated in the order π_1, π_2, \dots ; then on an instance $I \cdot s$, the values $h(x)$ would have to be enumerated in the different order $\pi_1 \cdot s, \pi_2 \cdot s, \dots$, which is not the case.

A similar phenomenon emerges in the polynomial-time algorithm for solving CSP over a template admitting a Malcev polymorphism, designed in [7] (or its generalized variant [10]). The algorithm is applicable to coset templates as they admit a Malcev polymorphism, defined as $\phi(x, y, z) = xy^{-1}z$ (whenever x, y, z are elements of the same carrier group). Roughly speaking, the algorithm manipulates a set of pre-solutions of an instance (succinctly represented by polynomially many representatives). For some fixed ordering c_1, \dots, c_n of all constraints in an instance, in its k th phase the algorithm computes the pre-solutions satisfying the constraints c_1, \dots, c_k . Even if the core operation performed by the algorithm, namely computation of the Malcev polymorphism ϕ , is equivariant, that is

$$\phi(x \cdot s, y \cdot s, z \cdot s) = \phi(x, y, z) \cdot s,$$

the whole algorithm is not so. As above, responsible for non-equivariance is enumeration of all elements of the template.

5. Local equivariant algorithm implies 2-Helly

Theorem 5.1. *For a coset template T , if $\text{CSP}(T)$ is solvable by a local equivariant algorithm then T is 2-Helly.*

In other words, no local equivariant algorithm can solve $\text{CSP}(T)$, when T a coset template but not 2-Helly. As a direct corollary, bounded width implies 2-Helly for coset templates. Note that the converse of Theorem 5.1 holds as well, as 2-Helly implies bounded width.

Another direct consequence of Theorem 5.1 is that coset templates T for which $\text{CSP}(T)$ is definable in LFP, IFP or IFP+C, are 2-Helly. In consequence, over coset templates, all the mentioned fix-point extensions of first-order logic are equally expressive, and equivalent to bounded width.

The rest of this section is devoted to the proof of Theorem 5.1: assuming a coset template T is not 2-Helly, we construct a family of instances that are hard for every local equivariant algorithm. Interestingly, the hard instances are a generalization of CFI graphs [8]. The idea of the proof generalizes the construction of [6].

Proof. Fix a coset template T being not 2-Helly, and a local equivariant algorithm. We aim at showing that the algorithm does not correctly solve $\text{CSP}(T)$. Let $\mathcal{D}_i(I)$ denote the object computed in the i th stage of the algorithm on input I .

We start with the following claim, whose proof is postponed to Section 5.1:

Proposition 5.2. *There are some subgroups S_1, S_2, S_3 of carrier groups, and an almost-direct product*

$$R \leq S_1 \times S_2 \times S_3$$

such that R and all its cosets in $S_1 \times S_2 \times S_3$ are pp-definable (as ternary relations).

Following the idea of [6], we will now define a class of instances, called n -torus instances, and then show that the consistency algorithm yields incorrect results for these instances. An n -torus instance is an instance of particular shape. It contains exactly $3n^2$ elements

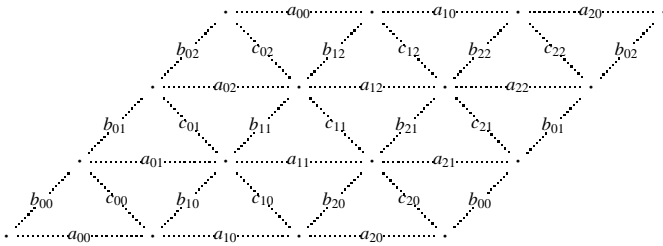
$$a_{ij}, b_{ij}, c_{ij}, \quad \text{for } i, j \in \{0 \dots n-1\},$$

and exactly $2n^2$ constraints:

$$R_{ij}(a_{ij}, b_{ij}, c_{ij}) \text{ and } R'_{ij}(a_{i(j+1)}, b_{(i+1)j}, c_{ij}), \quad (7)$$

for $i, j \in \{0 \dots n-1\}$. We adopt the convention that all indices are counted modulo n , e.g. $a_{in} = a_{i0}$ and $a_{ni} = a_{0i}$. Relations R_{ij} and R'_{ij} are arbitrary cosets of R in $S_1 \times S_2 \times S_3$; by Proposition 5.2 they are all pp-definable. Formally, the constraints in n -torus instance are not just relations from T , but rather pp-definable relations in T . We rely here on a folklore common knowledge: a pp-definable constraint can be simulated by adjoining to an instance a gadget, whose size is the number of (existentially) quantified variables in the defining pp-formula.

The $2n^2$ tuples (a_{ij}, b_{ij}, c_{ij}) and $(a_{i(j+1)}, b_{(i+1)j}, c_{ij})$ appearing in constraints (7) we call *positions* of an n -torus. The shape of a 3-torus instance is depicted below, with triangles representing positions and sides of triangles representing elements.



Every element a of an n -torus appears in exactly two constraints. Thus every position (a, b, c) has exactly three neighbors, namely those other positions that contain any of a, b, c .

For instance, neighbors of the position (a_{12}, b_{21}, c_{11}) are

$$(a_{12}, b_{12}, c_{12}), \quad (a_{21}, b_{21}, c_{21}), \quad \text{and } (a_{11}, b_{11}, c_{11}).$$

This defines the 3-regular neighborhood graph, with vertices being the positions of an n -torus.

The n -torus instances are built from triangulations of a torus surface. It is however not particularly important to use a torus; equally well a sphere could be used instead, or any other connected closed surface, as long as, intuitively speaking, the surface is hard to decompose into small pieces. The non-decomposability can be formally stated as follows:

Fact 5.3. *After removing $j < n$ positions, the neighborhood graph of an n -torus still contains a connected component of size at least $2n^2 - j^2$.*

Indeed, locally, the neighborhood graph of an n -torus can be seen as a 3-regular graph on the plane. Thus, in order to isolate j^2 positions one needs to cut more than j edges.

Definition 5.4. *Let I be an n -torus, and $i \geq 0$. We say that the algorithm ignores a position (a, b, c) of I after the i th stage, if*

$$\mathcal{D}_i(I) = \mathcal{D}_i(I')$$

for every n -torus I' that differs from I only by one constraint at position (a, b, c) .

Using Fact 5.3 we prove:

Proposition 5.5. *There is some $m \in \mathbb{N}$ such that for sufficiently large n and every n -torus I , the algorithm ignores, after every stage, all but at most m positions of I .*

Therefore for every sufficiently large instance, the algorithm necessarily ignores some position after the last stage, which easily entails incorrectness of the algorithm. Indeed, consider an n -torus I_R that uses the relation R of Proposition 5.2 in all its constraints. Being a subgroup instance, I_R is solvable and hence the algorithm answers positively. On one hand, by Proposition 5.5 there is some position (a_0, b_0, c_0) such that the algorithm answers positively for the instance obtained by replacing the relation R in the constraint $R(a_0, b_0, c_0)$ in I_R with any other coset of R in $S_1 \times S_2 \times S_3$. On the other hand, we prove:

Proposition 5.6. *Replacing the relation R with any other coset of R in one constraint in I_R yields an unsolvable instance.*

In consequence, the algorithm is incorrect. This completes the proof of Theorem 5.1, once we prove the three yet unproved claims, namely Propositions 5.2, 5.5 and 5.6. \square

5.1. Proof of Proposition 5.2

By Lemma 2.2 some instance I contains a $(k, k+1)$ -anomaly, for $k \geq 2$. Note that this implies that this instance has at least one solution.

Wlog. we can assume that I is a subgroup instance. Indeed, for an arbitrary solution h of I , define a new instance by the action of h^{-1} :

$$I' := I \cdot h^{-1}.$$

As h is a solution of I , for every constraint $R(a_1, \dots, a_n)$ in I , the tuple $(h(a_1), \dots, h(a_n))$ is in R . Hence every relation appearing in a constraint of I' is a subgroup in the product of constraining groups, as required. Due to Fact 3.5 an anomaly admitted by I translates to an anomaly admitted by I' .

By Lemma 3.4 we deduce that some (possibly different) instance I admits a $(2, 3)$ -anomaly $h = (\pi_1, \pi_2, \pi_3)$. Consider the set H of all those partial solutions, with the same domain as h , that extend to a solution of I . H is a pp-definable ternary relation according to Fact 2.3. By Fact 3.2(2) we know that H is a subgroup in the product $G_1 \times G_2 \times G_3$ of some three carrier groups. As h is a $(2, 3)$ -anomaly, we know (we prefer below to write $H(\pi_1, \pi_2, \pi_3)$ instead of $(\pi_1, \pi_2, \pi_3) \in H$):

$$\neg H(\pi_1, \pi_2, \pi_3) \quad (8)$$

$$\exists \tau \in G_1. H(\tau, \pi_2, \pi_3) \quad (9)$$

$$\exists \tau \in G_2. H(\pi_1, \tau, \pi_3) \quad (10)$$

$$\exists \tau \in G_3. H(\pi_1, \pi_2, \tau) \quad (11)$$

Now we are ready to define an almost-direct product $R \leq S_1 \times S_2 \times S_3$. The subgroups $S_1 \leq G_1$, $S_2 \leq G_2$ and $S_3 \leq G_3$ we define as follows:

$$\tau_1 \in S_1 \iff \exists \tau. H(\tau_1, \tau, 1) \wedge \exists \tau. H(\tau_1, 1, \tau)$$

$$\tau_2 \in S_2 \iff \exists \tau. H(\tau, \tau_2, 1) \wedge \exists \tau. H(1, \tau_2, \tau)$$

$$\tau_3 \in S_3 \iff \exists \tau. H(\tau, 1, \tau_3) \wedge \exists \tau. H(1, \tau, \tau_3)$$

and the subgroup R as the restriction of H to $S_1 \times S_2 \times S_3$:

$$R := H \cap S_1 \times S_2 \times S_3.$$

By the very definition, S_1, S_2, S_3 and R are pp-definable.

We need to show the conditions (2)–(5) in Definition 2.4. For (2) (i.e. $R \neq S_1 \times S_2 \times S_3$) we use (8) and (9) to conclude that for $\tau_1 = \tau^{-1}\pi_1 \in G_1$ it holds

$$\neg H(\tau_1, 1, 1).$$

Moreover, using (9) and (10) we deduce that for some $\bar{\tau} \in G_2$

$$H(\tau_1, \bar{\tau}, 1); \quad (12)$$

and similarly, using (9) and (11), we deduce that for some $\bar{\tau} \in G_3$,

$$H(\tau_1, 1, \bar{\tau}).$$

Thus $\tau_1 \in S_1$ and therefore $(\tau_1, 1, 1) \in S_1 \times S_2 \times S_3 \setminus R$.

Now we concentrate on condition (5) in Definition 2.4 (the remaining two conditions (3) and (4) are shown in the same way). Let $\tau_1 \in S_1$ and $\tau_2 \in S_2$. By the very definition of S_1 and S_2 we learn

$$H(\tau_1, 1, \tau) \quad (13)$$

$$H(1, \tau_2, \tau') \quad (14)$$

for some $\tau, \tau' \in G_3$. Therefore $H(\tau_1, \tau_2, \tau\tau')$ and it only remains to show that $\tau\tau' \in S_3$. Consider τ (τ' is treated analogously) in order to show $\tau \in S_3$. The fact (13) proves a half of the defining condition for $\tau \in S_3$, while the other half is proved by combining (13) with (12) to deduce $H(1, \bar{\tau}^{-1}, \tau)$. We have thus shown that R is an almost-direct product of S_1, S_2, S_3 .

Fact 3.3(2) guarantees that all cosets of R are pp-definable, as required.

5.2. Proof of Proposition 5.5

We will need the following property of almost-direct products:

Lemma 5.7. *Every coset R' of R in $S_1 \times S_2 \times S_3$ contains elements of the form*

$$(\tau_1, 1, 1), \quad (1, \tau_2, 1), \quad (1, 1, \tau_3),$$

for some $\tau_1 \in S_1, \tau_2 \in S_2$ and $\tau_3 \in S_3$.

Proof. Indeed, let $\pi = (\pi_1, \pi_2, \pi_3) \in R'$. Knowing that $\rho = (\rho_1, \pi_2, \pi_3) \in R$ for some $\rho_1 \in S_1$, we get

$$\rho^{-1}\pi = (\rho_1^{-1}\pi_1, 1, 1) \in R'$$

as required. Likewise one proves the remaining two claims. \square

Let $m = (2d)^2$, where d is the locality bound of the algorithm, and let I be an n -torus, for n sufficiently large to satisfy $2n^2 > (d+1) \cdot m$.

We proceed by induction on the number i of a stage. Observe that elements of a set $S \subseteq I$ of size at most d appear in at most $2d$ positions of I . Therefore, whenever (L0) applies (this is necessarily the case when $i = 1$), all but at most $2d < m$ positions are ignored after the i th stage.

Otherwise, suppose (L1) and (L2) hold. Let $S \subseteq I$ be such that

$$\mathcal{D}_i(I) \cdot s = \mathcal{D}_i(I) \quad \text{for all } s \text{ fixing } S. \quad (15)$$

By (L2) we can assume that the size of S is at most d . By the induction assumption for the previous stages and by (L1), there are at most $d \cdot m$ positions not ignored after stage i . We need to show, however, that there are at most m such positions.

The argument has a geometric flavor, and builds on Fact 5.3: after removing from the neighborhood graph all positions in which elements of S appear (there is at most $2d$ of them), there is a connected subset C of positions of size at least $2n^2 - m$, so it is larger than $d \cdot m$. By the induction assumption, some position in C is ignored after the i th stage. For the proof of Proposition 5.5 it is enough to prove that *every* position in C is ignored after i th stage. To this end, since C is connected, it is now enough to show:

Claim 5.7.1. *If some position in C is ignored after the i th stage, then every neighbor of that position in C also is.*

To show the last claim, consider two neighboring constraints in C , say $U(a, b, c)$ and $U'(a, b', c')$, both referring to an element a . Supposing that (a, b, c) is ignored, we need to demonstrate that (a, b', c') is ignored too. Let \vec{I}' be an n -torus obtained from I by replacing the constraint $U'(a, b', c')$ with $\overline{U}'(a, b', c')$, for some coset $\overline{U}' = U'\pi$. We need to show

$$\mathcal{D}_i(I) = \mathcal{D}_i(\vec{I}'). \quad (16)$$

Using Lemma 5.7 we may assume wlog. that $\pi = (\pi_1, 1, 1)$ for some $\pi_1 \in S_1$. Let s be a pre-solution defined by

$$s(x) = \begin{cases} \pi_1 & \text{if } x = a \\ 1 & \text{otherwise.} \end{cases}$$

Knowing that (a, b, c) is ignored, we may write

$$\mathcal{D}_i(I) = \mathcal{D}_i(\bar{I}), \quad (17)$$

where the n -torus \bar{I} is obtained from I by replacing the constraint $U(a, b, c)$ with $\bar{U}(a, b, c)$, for $\bar{U} = U\pi^{-1}$. Observe the equality

$$\bar{I}' = \bar{I} \cdot s. \quad (18)$$

Now we are ready to prove (16) by composing the following equalities:

$$\mathcal{D}_i(I) = \mathcal{D}_i(I) \cdot s = \mathcal{D}_i(I \cdot s) = \mathcal{D}_i(\bar{I} \cdot s) = \mathcal{D}_i(\bar{I}').$$

The first equality follows by (15), as s fixes \mathcal{S} ; the second one is the equivariance condition; the third one follows by (17) combined with equivariance; and the last one is a consequence of (18).

5.3. Proof of Proposition 5.6

Fix a position (a_0, b_0, c_0) . Let I_R^- be the instance obtained from I_R by removing the constraint $R(a_0, b_0, c_0)$. We will show that every solution h of I_R^- satisfies the constraint $R(a_0, b_0, c_0)$:

$$(h(a_0), h(b_0), h(c_0)) \in R. \quad (19)$$

According to the definition of n -torus, the positions of I_R split into two disjoint subsets, call them *negative* and *positive*, so that neighbors of a negative position are positive, and vice versa. Wlog. assume that (a_0, b_0, c_0) is negative. Consider the following expression (symbol \prod stands for the group operation in R , applied in an unspecified but irrelevant order):

$$\prod_{(a,b,c) \text{ negative}} (h(a), h(b), h(c))^{-1} \prod_{(a,b,c) \text{ positive}} (h(a), h(b), h(c)), \quad (20)$$

where (a, b, c) in the first subexpression ranges over all negative positions of I_R^- (hence (a_0, b_0, c_0) is omitted), and in the second subexpression over all positive ones. The expression (20) evaluates to some value (π_1, π_2, π_3) in R .

Let $[_] : R \rightarrow [R]$ be a surjective group homomorphism from R to a commutative group $[R]$, guaranteed jointly by Lemmas 2.5 and 2.6. Recall from Section 2.4 that the homomorphism $[_]$ is defined point-wise, namely $[(\tau_1, \tau_2, \tau_3)] = ([\tau_1], [\tau_2], [\tau_3])$. Apply $[_]$ to (20) to get an expression:

$$\prod_{(a,b,c) \text{ negative}} ([h(a)], [h(b)], [h(c)])^{-1} \prod_{(a,b,c) \text{ positive}} ([h(a)], [h(b)], [h(c)]). \quad (21)$$

Observe that $[h(a)]$ appears in (21) exactly once, for every $a \in I$ different from a_0, b_0, c_0 ; the same applies to the inverse $[h(a)]^{-1}$. Thus, as $[R]$ is commutative, every $[h(a)]$ together with its inverse cancels out. Moreover, $[h(a_0)]$, $[h(b_0)]$ and $[h(c_0)]$ also appear in (21) exactly once, while their inverses do not appear as the negative position (a_0, b_0, c_0) has been omitted. Therefore the expression (21) evaluates to $([h(a_0)], [h(b_0)], [h(c_0)])$. On

the other hand, (21) necessarily evaluates to $([\pi_1], [\pi_2], [\pi_3])$. Using the notation of Section 2.4 we can write:

$$h(a_0) \equiv_1 \pi_1 \quad h(b_0) \equiv_2 \pi_2 \quad h(c_0) \equiv_3 \pi_3.$$

Now we use the closure of R on the equivalences, cf. (6) in Section 2.4: as $(\pi_1, \pi_2, \pi_3) \in R$, we deduce $(h(a_0), h(b_0), h(c_0)) \in R$ as required. Proposition 5.6 is thus proved.

Remark: Splitting the positions into positive and negative ones, with one more positive than negative ones, resembles the property of ability to count of [14]. We believe that the proof can be modified to prove the equivalence: a coset template is not 2-Helly if and only if some its pp-definable extension has the ability to count. The latter property needs however to be slightly generalized to work in our setting, as the setting allows many different carrier groups. The equivalence is not a new result: it been shown recently for all templates in [4].

Acknowledgements. The author is grateful to Szymek Toruńczyk for proposing a simplified proof of Lemma 2.6. Moreover, thanks go to Bartek Klin, Ania Ochremiak, and Szymek Toruńczyk for long and fascinating discussions on CSP and its relationship to computation in sets with atoms. The author thanks also Luc Segoufin for encouraging me to write down this note. Finally, thanks go to the anonymous reviewers for their insightful and helpful remarks.

- [1] Albert Atserias, Andrei A. Bulatov, and Anuj Dawar. Affine systems of equations and counting infinitary logic. *Theor. Comput. Sci.*, 410(18):1666–1683, 2009.
- [2] Libor Barto. The collapse of the bounded width hierarchy. *Journal of Logic and Computation*, 2014.
- [3] Libor Barto and Marcin Kozik. Constraint satisfaction problems of bounded width. In *Proc. FOCS'09*, pages 595–603. IEEE Computer Society, 2009.
- [4] Libor Barto and Marcin Kozik. Constraint satisfaction problems solvable by local consistency methods. *J. ACM*, 61(1):3, 2014.
- [5] Mikołaj Bojańczyk, Bartek Klin, and Sławomir Lasota. Automata with group actions. In *Proc. LICS'11*, pages 355–364, 2011.
- [6] Mikołaj Bojańczyk, Bartek Klin, Sławomir Lasota, and Szymon Toruńczyk. Turing machines with atoms. In *Proc. LICS'13*, pages 183–192, 2013.
- [7] Andrei A. Bulatov and Víctor Dalmau. A simple algorithm for Malcev constraints. *SIAM J. Comput.*, 36(1):16–27, 2006.
- [8] Jin-yi Cai, Martin Fürer, and Neil Immerman. An optimal lower bound on the number of variables for graph identifications. *Combinatorica*, 12(4):389–410, 1992.
- [9] Gregory L. Cherlin, Gary A. Martin, and Daniel H. Saracino. Arities of permutation groups: Wreath products and k -sets. *Journal of Combinatorial Theory, Series A*, 74(2):249 – 286, 1996.
- [10] Víctor Dalmau. Generalized majority-minority operations are tractable. *Logical Methods in Computer Science*, 2(4), 2006.
- [11] Víctor Dalmau and Benoit Larose. Maltsev + Datalog \rightarrow symmetric Datalog. In *Procs. LICS*, pages 297–306. IEEE Computer Society, 2008.
- [12] H.D. Ebbinghaus and J. Flum. *Finite Model Theory*. Perspectives in mathematical logic. Springer, 1999.
- [13] Tomás Feder. Constraint satisfaction on finite groups with near subgroups. *Electronic Colloquium on Computational Complexity*, 005, 2005.
- [14] Tomás Feder and Moshe Y. Vardi. The computational structure of monotone monadic SNP and constraint satisfaction: A study through Datalog and group theory. *SIAM J. Comput.*, 28(1):57–104, 1998.
- [15] D. Hobby and R.N. McKenzie. *The Structure of Finite Algebras*, volume 76 of *Contemporary Mathematics*. AMS, Providence, R.I, 1988.
- [16] Benoit Larose and László Zádori. Bounded width problems and algebras. *Algebra Universalis*, 56(3):439–466, 2007.